PMM U.S.S.R.,Vol.45,pp.454-458
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0021-8928/82/4 0454 $7.50/0

UDC 531.38

QUALITATIVE INVESTIGATIONS OF PLANE AND SIMILAR MOTIONS OF A SOLID ABOUT A FIXED POINT*

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The plane motion of a solid about a fixed point in a conservative force field is investigated. The motion is defined by a single second order differential equation. Stability of plane motions is investigated in the first approximation. Similar to plane periodic motions of a solid are analyzed by the Poincaré method of small parameter.

The equations of motion of a solid about a fixed point in a conservative force field that admits the areas integral were reduced in /1/ to the differential equation

$$\frac{y''}{(1+y'^2)^{3/2}} = \pm\frac{\Omega}{\sqrt{2U}} + \frac{1}{2U}(U_y - U_x y')(1+y'^2)^{-1/2} \tag{1}$$

$$x = \sqrt{\frac{C}{A}}\int_0^\sigma \frac{d\sigma}{(1-n\sigma^2)\sqrt{(1-k^2\sigma^2)(1-\sigma^2)}},$$

$$y = \sqrt{\frac{A}{C}}\int_0^\rho \frac{d\rho}{(1+m\rho^2)\sqrt{(1-k'^2\rho^2)(1-\rho^2)}}$$

$$\Omega = \frac{f}{\sqrt{AC}}(1-k^2\sigma^2-k'^2\rho^2)\sqrt{(1-n\sigma^2)(1+m\rho^2)}\times$$
$$[A-B+C-2(A-B)\sigma^2+2(B-C)\rho^2]$$

$$U = B(1-k^2\sigma^2-k'^2\rho^2)\left[h+U_0-\frac{f^2}{2B}(1-n\sigma^2)(1+m\rho^2)\right]$$

$$k^2 = 1-k'^2 = \frac{A-B}{A-C},\quad n = \frac{A-B}{A},\quad m = \frac{B-C}{C}$$

where A, B, C are the principal moments of inertia of the body, h is Jacobi's constant, f is the areas constant, and σ and ρ are elliptic coordinates related to directional cosines of the field axis of symmetry with respect to the body principal axes of inertia by formulas

$$(\gamma_1, \gamma_2, \gamma_3) = \frac{1}{\sqrt{(1-n\sigma^2)(1+m\rho^2)}}\left(\sqrt{\frac{B}{A}}\sigma\sqrt{1-k'^2\rho^2},\right.$$
$$\left.\sqrt{(1-\sigma^2)(1-\rho^2)},\quad \sqrt{\frac{B}{C}}\rho\sqrt{1-k^2\sigma^2}\right)$$

and U_0 is the force function of the initial force field. We shall investigate the stability of one of the simple periodic motions of the body, namely, its plane motion.

The plane motion of the body is understood to be a motion at variable angular velocity about one of its principal axes of inertia which is constantly maintained in a horizontal position. Motion of the body is plane $y(x) = 0$ (about axis C) then, when the areas constant $f = 0$ and variations of the force function satisfy the condition $(\partial U/\partial y)_{y=0} \equiv 0$. The nature of motion depends on Jacobi's constant h.

Let us consider the orbital stability of the representative point trajectory in the plane xy, first, for a fixed h. If $f = 0$, Eq.(1) of that trajectory is of the form

$$2Uy'' + (U_x y' - U_y)(1+y'^2) = 0 \tag{2}$$

We shall investigate stability of the zero solution of Eqs.(2) in the first approximation.

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*Prikl.Matem.Mekhan.,45,No.4,618-623,1981

Expanding function U in series in powers of y and retaining in Eq.(2) only terms linear in y, y', we obtain

$$2U^\circ y'' + U_x^\circ y' - U_{yy}^\circ y = 0$$
$$U^\circ = U(x, 0),\quad U^\circ_x = U_x(x, 0),\quad U_{yy}^\circ = U_{yy}(x, y)|_{y=0}$$

or what is the same

$$\sqrt{U^\circ}\frac{d}{dx}\left(\sqrt{U^\circ}\frac{dy}{dx}\right) - \frac{1}{2}U^\circ_{yy}y = 0$$

Introducing the new variable u by formula $du = dx/\sqrt{U^\circ}$, we reduce the last equation to the form

$$\frac{d^2y}{du^2} + p(u)\,y = 0, \quad p(u) = -\frac{1}{2}U^\circ_{yy}(x(u)) \tag{3}$$

It can be shown that for all admissible values of Jacobi's constant except the bifurcational, $p(u)$ is a periodic function of period

$$T = \oint \frac{dx}{\sqrt{U^\circ}}$$

where the integral is taken over the interval that corresponds to one turn in the rotary motion (or to a complete oscillation in the case of libration).

The solid body can perform plane motions $y(x) = 0$ in a Newtonian gravitation field when its center of mass lies in the equatorial plane of the central ellipsoid of inertia. We then have

$$U = (1 - k^2\sigma^2 - k'^2\rho^2)\left[h - \frac{a\sqrt{\frac{B}{A}}\,\sigma\sqrt{1-k'^2\rho^2} + b\sqrt{(1-\sigma^2)(1-\rho^2)}}{\sqrt{(1-n\sigma^2)(1+m\rho^2)}} - \frac{3g}{2BR(1-n\sigma^2)(1+m\rho^2)}\right] \tag{4}$$

$$p = \frac{C}{A}\left\{k'^2\left(h + \frac{3gA}{2BCR} - k'^2 a\sqrt{\frac{B}{A}}\frac{\sigma}{\sqrt{1-n\sigma^2}}\times\right.\right.$$
$$\left.\left.\left[\frac{A+2C}{2C} - \frac{Ak^2}{2C}\sigma^2\right] - b\sqrt{\frac{1-\sigma^2}{1-n\sigma^2}}\left[k'^2 + \frac{B}{C} - \frac{Bk^2}{2C}\sigma^2\right]\right\}$$

where a, b are the products of the body mass by the coordinates of the center of mass relative to the principal planes, and g is the gravitation force at distance R from the center.

Stability of the zero solution of Eq.(2) in the first approximation can be assessed by applying to Hill's equation (3) the criterion of boundedness of solutions of linear differential equations with periodic coefficients /2/.

Let us also consider some simple cases of motion in a homogeneous gravitational field.

1°. $A = B$, $b = 0$.

a) Rotary motion $(h > a)$. We have

$$\sigma = -1 + 2\,\mathrm{sn}^2(u, \nu), \quad \nu^2 = \frac{2a}{h+a}, \quad p = \frac{4C^2}{A^2}\left(1 + \frac{A\nu^2}{4C}\right) - \frac{2C(A+2C)}{A^2}\nu^2\,\mathrm{sn}^2 u$$

Using the notation $\alpha = 2C/A$ we reduce Eq.(3) to the form of Lamé equation

$$\frac{d^2y}{du^2} + \left[\alpha\left(\alpha + \frac{\nu^2}{2}\right) - \alpha(\alpha+1)\nu^2\,\mathrm{sn}^2 u\right]y = 0 \tag{5}$$

We shall try to elucidate the pattern of stability and instability in the plane of parameters $\alpha\nu$. As h increases from α to ∞ at fixed C/A, ν decreases from 1 to 0 under the condition $0 < C \leqslant B = A$, $0 < \alpha \leqslant 2$. It was shown in /2/ that the domains of stability $\mathbf{O}_s$ and instability $\mathbf{H}_s$ are separated by lines Π_s^+, Π_s^- whose equations define the distribution of eigenvalues of the boundary value problem of periods $2K, 4K$. We have, for instance, /3/

$$\alpha(2-\nu^2) + \frac{(\alpha^2-1)(\alpha+2)\nu^4/8}{(2-\nu^2)(1-\alpha^2/4)} - \frac{(\alpha^2-9)(\alpha-2)(\alpha+4)\nu^4/16^2}{(2-\nu^2)(1-\alpha^2/16)} - \frac{(\alpha^2-25)(\alpha+4)(\alpha+6)\nu^4/48}{(2-\nu^2)(1-\alpha^2/36)} - \ldots = 0 \tag{6}$$

for Π_{2s}^+ and

$$(\alpha-2)(\alpha+2)(2-\nu^2) + \frac{(\alpha-2)(\alpha^2-9)(\alpha+4)\nu^4/8^2}{(2-\nu^2)(1-\alpha^2/16)} - \frac{(\alpha-4)(\alpha^2-25)(\alpha+6)\nu^4/48^2}{(2-\nu^2)(1-\alpha^2/36)} - \ldots = 0 \tag{7}$$

for Π_{2s}^-. Curves $\Pi_s\pm$ pass through point $(s, 0)$. Equation (7) implies that $\alpha = 2$ belongs to Π_2^-. It can be also shown /4/ that curves $\Pi_s\pm$ have at point $(s, 0)$ a contact of not less than first order with the line $\alpha = s$. The stability domain $\mathbf{O}_s$ $(s = 0, 1)$ contains on axis α the segment $s < \alpha \leqslant s + 1$. These properties are shown in Fig.1. The following cases have a particular effect on stability pattern.

Kovalevska's top $(\alpha = 1)$. Plane rotation of such body is stable in the limit case of $\nu = 0\,(h = \infty)$, while instability occurs at finite but arbitrarily large values of h.

The case of $A = 4C$. We shall show that for this body the representative point $(^1/_2, \nu)$ always belongs to $\mathbf{O}_0$. Indeed, Eq.(5) assumes the form

$$\frac{d^2y}{du^2} + \left(\frac{1+\nu^2}{4} - \frac{3}{4}\nu^2 \operatorname{sn}^2 u\right) y = 0 \tag{8}$$

and for $0 \leqslant \nu < 1$ has the limited general solution

$$y = \left(C_1 \operatorname{cn}\frac{u}{2} \operatorname{dn}\frac{u}{2} + C_2 \operatorname{sn}\frac{u}{2}\right)\left[1 - \nu^2 \operatorname{sn}^2\frac{u}{2}\right]^{-1/2}$$

where C_1, C_2 are arbitrary constants.

(b) Libration motion $h < a$. In this case

$$p(u) = \alpha\left(\frac{1}{2} + \alpha\nu^2\right) - \alpha(\alpha+1)\nu^2 \operatorname{sn}^2 u, \quad \nu^2 = \frac{h+a}{2a}, \quad \alpha = \frac{2C}{A}$$

When $0 < C \leqslant A$ and ν is small we have a single stability domain $\mathbf{O}_0$ which contains the segment $0 < \alpha \leqslant 2$. Its right-hand boundary Π_1 is tangent to line $\alpha = 2$ at point $\nu = 0$. Unlike the case of rotary motion the Kovalevska top has no resonance properties at small oscillations near the lower equilibrium position. The Goriachev—Chaplygin top retains that property, since in both cases, that of rotary and libration motions Eq.(8) represents the first approximation.

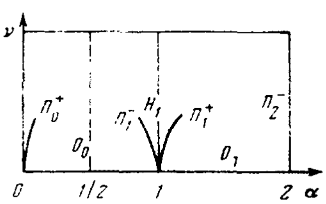

Fig.1

2^{O}. $B = C$, $a = b = 0$. To investiage the stability of trajectory $x = 0$ it is sufficient to carry out the substitution $a \to c$, $A \leftrightarrow C$ in equations of Case 1^{O}.

We obtain

a) In the case of rotary motion with $s < \alpha < s + 1$, $s = 2, 3$ and reasonably large h point (α, ν) belongs to the s-th stability domain $\mathbf{O}_s$. When $A = {}^3/_2\,C$, $A = 2C$ motion is stable in the limit case of rotation by inertia about axis A, but stability is lost for any aribtrarily small distance between the center of mass and the support point.

b) In the case of libration motion small plane oscillations about the lower equilibrium position are stable in the absence total dynamic stability.

3^{O}. Let us now consider the motion $y = 0$ of a dynamically nonsymmetric body $A > B > C$, $c = 0$. We apply the Joukowski criterion /2/. The motion belongs to the s-th stability domain when condition

$$s^2\pi^2 < T^2 p(u) \leqslant (s+1)^2\pi^2$$

where s is an integer, is satisfied.

In the case of rapid rotation we have two stability domains

$$s < 2\left(1 - \sqrt{\frac{(A-C)(B-C)}{AB}}\right) + o\left(\frac{\sqrt{a^2+b^2}}{h}\right) < s + 1, \quad s = 0, 1$$

The motion is stable for all admissible points of the plane $(B/A, C/A)$, except the point in the neighborhood of curves $C = 0, B = C, 4(A-C)(B-C) = AB$. A similar result is obtained in the case of rapid rotation about axis A. Curves $A = B + C$, $A = B$, $4(A-C)\cdot(A-B) = BC$ are exceptional.

So far we dealt with conditional stability, i.e. stability of trajectories at fairly small perturbations of initial conditions which do not affect h and the zero area constant. In fact the first requirement is immaterial, since the trajectory remains stable at such perturbations of Jacobi's constant for which point (α, ν) remains in the initial stability domain.

In the instability domains $\mathbf{H}_s$ of the linear equation (3) the zero solution of the nonlinear equation (2) is unstable. To establish the stability of zero solution of the nonlinear equation in domains $\mathbf{O}_s$ it is necessary to consider the effect of higher order terms.

Let us now pass to the proof of existence of a denumerable set of classes of almost plane periodic rotations of a heavy solid about a fixed point. We assume that the solid differs only slightly from a dynamically symmetric body and that its center of mass is fairly close to the principal axis of inertia. We take $y = 0$ $(p = q = \gamma_3 = 0)$ with $A = B$, $b = c = 0$, $f = 0$ as the generating solution, and introduce the small parameter μ in such a way that b, c, f, $k = [(A - B)(A - C)]^{1/2}$ vanish as μ approaches zero. We set

$$b/b_1 = c/c_1 = f/f_1 = k = \mu$$

where b_1, c_1, f_1 are finite constants.

We apply the Poincaré method of the small parameter for deriving a periodic solution of Eq. (1) in the form of series

$$y = \sum_{s=1}^{\infty} \mu^s y_s \tag{9}$$

which becomes trivial when $\mu = 0$.

Substituting (9) into Eq. (1.1) we obtain for $\{y_s\}$ the system of equations

$$\sqrt{U_0}\frac{d}{dx}\left(\sqrt{U_0}\frac{dy_s}{dx}\right) - \frac{1}{2}U^\circ_{yy}y_s = P_s \quad (s = 1, 2, \ldots) \tag{10}$$

where $\{P_s\}$ are algebraic functions of σ and polynomials in $y_{s-1}, \ldots, y_1, y_{s-1}', \ldots, y_1'$, and the zero subscript indicates the value of function for $\mu = 0$.

As shown in /2/, system (10) after a suitable substitution of the independent variable, reduces to a system of uniform inhomogeneous Lamé equations

$$\frac{d^2y_s}{du^2} + \left[\alpha\left(\alpha + \frac{\nu^2}{2}\right) - \alpha(\alpha + 1)\nu^2 \operatorname{sn}^2(u, \nu)\right] y_s = P_s \tag{11}$$
$$\alpha = 2C/A, \quad \nu^2 = 2a/(h + a)$$

where the coefficients at y_s and functions P_s are periodic functions of u of period $2K(\nu)$. If with $s = 1$ it is possible to select the constants of integration so as to have the solution unique and periodic of period $2qK$ (q is an integer), it is then possible to construct in a unique manner all remaining y_s $(s = 2, 3, \ldots)$ of the same period.

Such procedure can be always carried out in the case when the equation in variations does not admit even a single periodic solution of the same period as that of the sought solution /5/ or, what is the same, when the pair of numbers (α, ν) does not satisfy the equation of eigenvalues for the Lamé function of period $2qK$.

The last equation was investigated in somewhat greater detail /6,7/ for $q = 1, 2, 4$, and generally defines curve γ_q in the domain of variation of parameters $0 < \alpha \leqslant 2$, $0 \leqslant \nu \leqslant 1$. The combination of curves γ_q $(q = 1, 2, \ldots)$ constitutes in this domain a set of measure zero. This proves that for almost all C/A, $(h > a)$ there exists a denumerable set of classes of periodic solutions of Eq. (1) of period of the form $2q$ (q is an integer) which reduces to solution $y = 0$, and when $\mu = 0$ are absolutely convergent for fairly small μ. These classes are generally different. The trajectory of period $2qK$ closes on the ellipsoid of inertia after q $(q/2)$ turns when q is odd (even).

Examples. 1°. Periodic trajectories of period $2K$. The equations of eigenvalues were given in Sect.1. If parameters α, ν do not satisfy any of these equations, there exists a class of periodic solutions of period $2K$ of the indicated type.

2°. The case of $A = B = 4C$. For the generating solution, as shown above, the general solution of the equation in variations is periodic of period $8K$. Three classes of periodic solutions of periods $2K$, $4K$, $6K$ exist. Solutions of higher period are consistent with these solutions by virtue of periodicity of Eq. (11) and of the general solution of the equation in variations. This reasoning is insufficient for assessing the existence in this case of solutions of period $8K$.

For the generating solution $x = 0$ $(a = b = 0, f = 0)$ to which corresponds rotation about the highest inertia axis A we have a system similar to (11) in x_s with parameter $\alpha = 2A/C$.

To the obtained solutions corresponds a periodic variation of Euler's angles θ, φ and, generally, an almost periodic motion relative to the precession angle.

The author thanks V.G. Demin for his assistance in this work.

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Translated by J.J.D.